ON THE HILBERT SCHEME OF DETERMINANTAL SUBVARIETIES

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Abstract. We compute the dimension of the Hilbert scheme of subvarieties of positive dimension in projective space which are defined as vanishing loci of maximal minors of a matrix with polynomial entries.

1. Introduction

A determinantal subvariety $X \subset \mathbb{P}^n$ is the locus defined by the vanishing of all minors of maximal order of a given matrix M of homogeneous polynomials.Many classical varieties can be constructed in this way, for instance Segre and Veronese varieties, rational normal scrolls, Palatini scrolls, and so forth. Several authors have given important contributions ever since, let us refer to the monographs [Nor76], [BV88], [Wey03], [MR08] for extended reading.

If one desires to parametrize all determinantal varieties of a given type (i.e. for fixed degrees of the entries of M), one way is look at $[X]$ as a point of a component $\mathcal{H}$ of the Hilbert scheme of subscheme in $\mathbb{P}^n$, and study to what extent the family of determinantal varieties fills in $\mathcal{H}$. In this spirit, Ellingsrud proved in [Ell75] that determinantal varieties of codimension 2 and dimension ≥ 1 are unobstructed and their family is open and dense in $\mathcal{H}$. In the series of papers [KMMR$^+$01, KMR05, KM09], the same behavior was established in many more cases, leading to conjecture that this phenomenon should be general.

In this paper we show that the family of determinantal varieties fills in an open dense subset of a generically smooth irreducible component of the Hilbert scheme, provided that both the dimension and the codimension of X are at least 2. For $\dim(X) = 1$, we also compute the dimension of this family, that need not be open in an irreducible component of the Hilbert scheme.

To better state the result we adopt now a more precise language. Let V be a $(n+1)$-dimensional vector space over an algebraically closed field $\mathbf{k}$ and $\mathbb{P}^n = \mathbb{P}^n(V)$ be the associated projective space. Let $a \geq b$ be integers and let $\alpha_1 \leq \cdots \leq \alpha_a$ and

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$\beta_1 \leq \cdots \leq \beta_b$ be two non-decreasing sequences of integers of length a and b. We define:

$$\mathscr{A} = \bigoplus_{j=1,\dots,a} \mathscr{O}_{\mathbb{P}^n}(-\alpha_j), \qquad \mathscr{B} = \bigoplus_{i=1,\dots,b} \mathscr{O}_{\mathbb{P}^n}(-\beta_i).$$

Let ϕ be a morphism:

$$\phi : \mathscr{A} \to \mathscr{B}.$$

This morpshism ϕ can be represented by a matrix M_ϕ whose entries are homogeneous polynomials degree $\alpha_j - \beta_i$. The *determinantal variety* $X_\phi \subset \mathbb{P}^n$ is given by the degeneracy locus $\mathrm{D}_{b-1}(\phi)$, where the rank of ϕ is at most to $b-1$:

$$X_\phi = \mathrm{D}_{b-1}(\phi),$$

i.e. X_ϕ is cut in $\mathbb{P}^n$ by the $b \times b$ minors of the matrix representing ϕ. We assume that X_ϕ is not empty and has the expected codimension, namely:

$$c := \operatorname{codim}(X, \mathbb{P}^n) = a - b + 1.$$

This happens for general ϕ if and only if:

$$\alpha_i \geq \beta_i, \qquad \text{for all } i = 1, \dots, b, \qquad \text{and} \qquad \alpha_i > \beta_i, \qquad \text{for some } i = 1, \dots, b. \tag{1.1}$$

Let now $p(t)$ be the Hilbert polynomial of X_ϕ. Consider the Hilbert scheme $\mathrm{Hilb}_{p(t)}(\mathbb{P}^n)$ of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$, and let $\mathcal{H}$ be the irreducible component (or their union if there are many) of $\mathrm{Hilb}_{p(t)}(\mathbb{P}^n)$ containing the class of X_ϕ in $\mathcal{H}$, denoted by $[X_\phi]$.

If we want to parametrize the schemes X_ϕ arising this way, we are lead to introduce the vector space:

$$\mathbf{W} = \mathrm{Hom}_{\mathbb{P}^n}(\mathscr{A}, \mathscr{B}).$$

Further, we consider the algebraic group:

$$\mathbf{G} = \mathrm{Aut}_X(\mathscr{A}) \times \mathrm{Aut}_X(\mathscr{B}).$$

An element ρ of $\mathbf{G}$ consists of a pair (g, h), and acts on $\mathbf{W}$ by:

$$\rho.\phi = g \circ \phi \circ h^{-1}.$$

A quotient of an open subset of $\mathbf{W}$ is a generically smooth irreducible variety that we denote by $\mathcal{Y}$. We have then the natural rational map:

$$\begin{aligned} F : \mathcal{Y} &\dashrightarrow \mathcal{H} \\ [\phi] &\mapsto [X_\phi]. \end{aligned}$$

We state now the main result of this note. Note that $\dim(X_\phi) = n + b - a - 1$.

Theorem. *Choose integers n, $b \leq a-1$, $\alpha_1 \leq \cdots \leq \alpha_a$ and $\beta_1 \leq \cdots \leq \beta_b$, with:*

$$\alpha_i \geq \beta_{i+1}, \quad \forall i = 1, \dots, b-1, \quad \textit{and} \quad \alpha_i > \beta_i, \quad \textit{for some } i = 1, \dots, b. \tag{1.2}$$

i) If $n+b-a-1 \geq 1$, then the map F is generically finite, so $\dim(\mathrm{Im}(F)) = \dim(\mathcal{Y})$.
ii) Moreover, if $n+b-a-1 \geq 2$, then F is also dominant, in particular $\mathcal{H}$ is an irreducible, generically smooth variety and $\dim(\mathcal{H}) = \dim(\mathcal{Y})$.
iii) Finally, if $\beta_b < \alpha_1$, and $n+b-a-1 \geq 2$, then F is birational.

As mentioned above, our result was motivated by a conjecture of Kleppe and Miró-Roig, see [KM09, Conjecture 4.2], which is solved by part (ii) above. What we show is in fact stronger since (iii) proves uniqueness of determinantal representations in the range $\alpha_1 > \beta_b$. Further, part (i) above addresses the positive-dimensional range of [KM09, Conjectures 4.1]. Our result thus completes [KMR05, KM09, Kle09], where these conjectures are addressed for several ranges of the α_j's and β_i's.

One should be aware that the dimension $\dim(\mathcal{Y})$ can be calculated explicitly in terms of some binomial coefficients involving the α_j, β_i's and n. Indeed we set $c = a - b + 1$, and, according to [KM09], we define:

$$\lambda_c = \sum_{\substack{j=1,\dots,a\\ i=1,\dots,b}} \binom{\alpha_j - \beta_i + n}{n} + \binom{\beta_i - \alpha_j + n}{n} - \sum_{i,j=1,\dots,a} \binom{\alpha_i - \alpha_j + n}{n} - \sum_{i,j=1,\dots,b} \binom{\beta_i - \beta_j + n}{n} + 1. \tag{1.3}$$

Further we define, for $i = 3, \dots, c$ the integers:

$$\ell_i = \sum_{j=1,\dots,b+i-1} \alpha_j - \sum_{i=1,\dots,b} \beta_i, \qquad h_{i-3} = 2\alpha_{b+i-1} - \ell_i + n. \tag{1.4}$$

and finally, for $i = 0, \dots, c-3$, the integers:

$$K_{i+3} = \sum_{\substack{r+s=i\\ r,s\geq 0}} \sum_{\substack{1\leq i_1<\cdots<i_r\leq b+i+1,\\ 1\leq j_1\leq\cdots\leq j_s\leq b}} (-1)^{i-r} \binom{h_i + \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots + \beta_{j_s}}{n} \tag{1.5}$$

In these terms, the dimension of $\mathcal{Y}$ is:

$$\dim(\mathcal{Y}) = \lambda_c + K_3 + \cdots + K_c.$$

Then the theorem says that the closure of $\mathscr{H}$ in $\mathrm{Hilb}_{p(t)}(\mathbb{P}^n)$ is an irreducible variety of dimension $\lambda_c + K_3 + \cdots + K_c$, and in fact an irreducible component of $\mathrm{Hilb}_{p(t)}(\mathbb{P}^n)$ if $\dim(X_\phi) = n + b - a - 1 \geq 2$.

The above formula for $\dim(\mathcal{Y})$ becomes particularly readable for homogeneous matrices, as we point out in the next corollary, that settles the positive-dimensional range of [Kle09, Conjecture 3.2].

Corollary. *Let $b \leq a-1$, $d \geq 1$ be integers and set $\alpha_j = d, \beta_i = 0$ for all i, j, and assume $\dim(X_\phi) \geq 2$, i.e. $n + b - a - 1 \geq 2$. Then the map F is birational. In particular, $\mathcal{H}$ is an irreducible, generically smooth variety of dimension:*

$$\dim(\mathcal{H}) = ab\binom{n+d}{n} - a^2 - b^2 + 1.$$

Finally, let us only mention the resemblance of the theorem above with the result of [FF10], where the Hilbert scheme of Palatini is described in terms of a Grassmann variety.

Remark. Shortly before submitting our paper we learned of a preprint by Kleppe [Kle10], addressing very similar questions.

1.1. **Structure of the paper.** In the Section 2 we set up the framework needed to prove our results. First we recall the dimension of the variety $\mathcal{Y}$ (Section 2.1). We will mainly use the cokernel sheaf attached to ϕ, introduced in Section 2.2, and the fact that X_ϕ is the birational image of a complete intersection Y_ϕ in the projective bundle $\mathbb{P}(\mathscr{B})$ (see Section 2.3). This allows to calculate the canonical class of X_ϕ and Y_ϕ, which we do in Section 2.4. In Section 3 we calculate the sections of the normal bundle of X_ϕ. Section 4 contains our main lemmas on the fibers of the map F, and the proof of the main result. Finally, Section 5 is devoted to some remarks on the geometry of X_ϕ, we will comment briefly determinantal surfaces (see 5.1) and determinantal hypersurfaces (see 5.2).

1.2. **Notations and conventions.** Let $\mathbf{k}$ be an algebraically closed field, $n \geq 2$ be an integer and let V be an $(n+1)$-dimensional vector space over $\mathbf{k}$. We consider the projective space $\mathbb{P}^n$ of 1-dimensional quotients of V. Under this convention we have $\mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \cong V$. The quotient associated to an element $0 \neq \xi \in V^*$ is denoted by $[\xi]$. If Z is a subvariety of $\mathbb{P}^n$, the symbol H_Z will often denote the restriction of $c_1(\mathscr{O}_{\mathbb{P}^n}(1))$ to Z, or more generally $f^*(H_{\mathbb{P}^n})$ if f is a morphism from Z to $\mathbb{P}^n$.

Given a morphism ϕ of vector bundles on a given variety Z, we will denote by $\mathrm{D}_k(\phi)$ the locus consisting of the points z of Z such that ϕ_z has rank at most k (here ϕ_z is the evaluation of ϕ at z). The subscheme $\mathrm{D}_k(\phi)$ is defined by the vanishing of all $(k+1)$-minors of a matrix defining ϕ. Given a global section s of a vector bundle on Z we let $\mathbb{V}(s)$ be the vanishing locus of s.

If $\mathscr{E}$ is a vector bundle on Z, we will denote by $\mathscr{E}_z$ the vector space lying over $z \in Z$. If $Z = \mathbb{P}^n$ and $0 \neq \xi \in V^*$, we will freely write $\mathscr{E}_\xi$ for $\mathscr{E}_{[\xi]}$. We write ω_Z for the dualizing sheaf of a Cohen-Macaulay scheme Z. When Z' is a subvariety of Z and $\mathscr{E}$ is a sheaf on Z, we will denote by $\mathscr{E}_{Z'}$ the restriction of $\mathscr{E}$ to Z'. The same notation will be used for the restriction of divisor classes. We will denote by $\mathcal{I}_{Z',Z}$ (or simply by $\mathcal{I}_{Z'}$) the ideal sheaf of Z' in Z and by $\mathscr{N}_{Z',Z}$ the normal sheaf of Z' in Z. We will denote by $S^k\mathscr{E}$ and $\wedge^k\mathscr{E}$ the k-th symmetric and alternating power of $\mathscr{E}$.

We refer to [MR08] and [BV88] for general material on determinantal subvarieties.

2. Basic constructions

Let us recall the notation from the introduction and keep it in mind throughout the paper. Given integers $b \leq a-1$, and $\alpha_1 \leq \cdots \leq \alpha_a$ and $\beta_1 \leq \cdots \leq \beta_b$, we will write:

$$\mathscr{A} = \bigoplus_{j=1,\ldots,a} \mathscr{O}_{\mathbb{P}^n}(-\alpha_j), \qquad \mathscr{B} = \bigoplus_{i=1,\ldots,b} \mathscr{O}_{\mathbb{P}^n}(-\beta_i).$$

The letter ϕ will always denote a morphism:

$$\phi : \mathscr{A} \to \mathscr{B}.$$

And we will write X_ϕ the first degeneracy locus of ϕ, so $X_\phi = \mathrm{D}_{b-1}(\phi)$, with the scheme-theoretic structure given by the $b \times b$ minors of ϕ.

2.1. **The map F.** Let $\mathscr{A}$ and $\mathscr{B}$ as above. The following lemma is a summary of well-known facts concerning the $\mathbf{G}$-orbit space of $\mathbf{W}$.

Lemma 2.1. *There is a generically smooth irreducible variety $\mathcal{Y}$ parametrizing generic $\mathbf{G}$-orbits of $\mathbf{W}$, with:*

$$\dim(\mathcal{Y}) = \lambda_c + K_3 + \cdots + K_c. \tag{2.1}$$

Proof. The group $\mathbf{G}$ is in general non-reductive. However, we will only be interested in some open piece of the orbit space. According to a result of Rosenlicht, [Ros63], there is a dense open subset $\mathbf{W}^\circ$ such that the quotient $\mathbf{W}^\circ/\mathbf{G}$ is geometric. Let us denote:

$$\mathcal{Y} = \mathbf{W}^\circ/\mathbf{G}.$$

This is a generically smooth variety of dimension:

$$\dim(\mathbf{W}) - \dim(\mathbf{G}) + \mathbf{G}_\phi,$$

where $\mathbf{G}_\phi$ is the stabilizer of a general element $\phi \in \mathbf{W}^\circ$. The dimension of $\mathbf{G}_\phi$ is computed in [KMMR+01, KMR05], and we get (2.1). □

2.2. **Cokernel sheaf on degeneracy loci.** Here we consider $\phi : \mathscr{A} \to \mathscr{B}$ as above, and we assume that $X = X_\phi$ has codimension $c = a - b + 1$. Then we have:

$$\dim(X) = n + b - a - 1.$$

We define the sheaf:

$$\mathscr{C}_\phi = \operatorname{coker}(\phi).$$

Note that $\mathscr{C}_\phi$ is supported on X_ϕ. Let i denote the embedding of X_ϕ in $\mathbb{P}^n$. Then $\mathscr{C}_\phi \cong i_*(\mathscr{L}_\phi)$, for a sheaf $\mathscr{L}_\phi$ on X_ϕ of (generic) rank 1. The sheaf $\mathscr{L}_\phi$ is ACM on X_ϕ, in particular it is reflexive, hence invertible if X_ϕ is integral and locally factorial.

Further, we have an exact sequence:

$$0 \to \mathscr{K}_\phi \to \mathscr{A} \to \mathscr{B} \to \mathscr{C}_\phi \to 0, \tag{2.2}$$

where we have set $\mathscr{K}_\phi = \ker(\phi)$. By [KMR05, Lemma 3.2], we have:

$$\mathscr{H}om_{\mathbb{P}^n}(\mathscr{C}_\phi, \mathscr{C}_\phi) \cong \mathscr{O}_{X_\phi}.$$

Therefore, applying $\mathscr{H}om_{\mathbb{P}^n}(-, \mathscr{C}_\phi)$ to (2.2), we obtain:

$$0 \to \mathscr{O}_{X_\phi} \to \mathscr{B}^* \otimes \mathscr{C}_\phi \xrightarrow{\psi} \mathscr{A}^* \otimes \mathscr{C}_\phi \to \mathscr{F}_\phi \to 0, \tag{2.3}$$

where the sheaf $\mathscr{F}_\phi$, defined by the exact sequence above, is supported on X_ϕ. We have:

$$\mathscr{E}xt^1_{\mathbb{P}^n}(\mathscr{C}_\phi, \mathscr{C}_\phi) \subset \mathscr{F}_\phi. \tag{2.4}$$

These exact sequences will play an important role in the computation of the normal sheaf of X_ϕ in $\mathbb{P}^n$.

Given the sheaf $\mathscr{L}_\phi$, we will also consider $c_1(\mathscr{L}_\phi)$, as a divisor class in $\operatorname{Cl}(X)$ by looking at the zero locus D_ϕ of $\mathscr{L}_\phi(t)$ (where we choose the smallest $t \in \mathbb{Z}$ such that $\mathrm{H}^0(X_\phi, \mathscr{L}_\phi(t)) \neq 0$). This locus is in fact a determinantal subvariety $X_{\phi_0} \subset \mathbb{P}^n$ with

$\phi_0 : \mathscr{A} \to \mathscr{B}_0$ and $\mathscr{B}_0 = \mathscr{B}/\mathscr{O}_{\mathbb{P}^n}(-t)$. Note also that the class of D_ϕ determines $\mathscr{L}_\phi$, because setting $\mathscr{L}_\phi^* = \mathscr{H}om_{X_\phi}(\mathscr{L}_\phi, \mathscr{O}_X)$, we have:

$$0 \to \mathscr{L}_\phi^*(-t) \to \mathscr{O}_{X_\phi} \to \mathscr{O}_{D_\phi} \to 0,$$

and this gives back $\mathscr{L}_\phi$ since $\mathscr{L}_\phi^{**} \cong \mathscr{L}_\phi$.

2.3. **Determinantal subvarieties as complete intersections.** Consider the vector bundle $\mathscr{B} = \bigoplus_{i=1,\dots,b} \mathscr{O}_{\mathbb{P}^n}(-\beta_i)$ over the projective space $\mathbb{P}^n$ and define the projective bundle:

$$\mathcal{P} = \mathbb{P}(\mathscr{B}) \xrightarrow{\pi} \mathbb{P}^n.$$

We have the relatively ample line bundle $\mathscr{O}_{\mathscr{B}}(1)$ and we let:

$$P = c_1(\mathscr{O}_{\mathscr{B}}(1)).$$

Set $\mathcal{T}_{\mathscr{B}}$ for the relative tangent bundle. We have:

$$0 \to \mathscr{O}_{\mathcal{P}}(-P) \to \pi^*(\mathscr{B}^*) \to \mathcal{T}_{\mathscr{B}}(-P) \to 0, \qquad \pi_*(\mathscr{O}_{\mathcal{P}}(P)) \cong \mathscr{B}, \tag{2.5}$$

Let us now describe the way of passing from a determinantal subvariety to a complete intersection in the projective bundle $\mathcal{P}$. There are isomorphisms:

$$\mathbf{W} = \mathrm{Hom}_{\mathbb{P}^n}(\mathscr{A}, \mathscr{B}) \cong \mathrm{Hom}_{\mathcal{P}}(\pi^*(\mathscr{A}), \mathscr{O}_{\mathcal{P}}(P)) \cong \bigoplus_{\substack{j=1,\dots,a \\ i=1,\dots,b}} S^{\alpha_j - \beta_i} V$$

Therefore, to an element ϕ of $\mathbf{W}$ there corresponds a section:

$$s_\phi \in \mathrm{H}^0(\mathcal{P}, \pi^*(\mathscr{A})^*(P)),$$

and thus a subvariety:

$$Y_\phi = \mathbb{V}(s_\phi) \subset \mathcal{P}.$$

The next lemma relates Y_ϕ to X_ϕ. To see how, we recall that $\mathrm{coker}(\phi) \cong i_*(\mathscr{L}_\phi)$, for a rank-1 sheaf $\mathscr{L}_\phi$ on X_ϕ. We consider the scheme $\mathbb{P}(\mathscr{L}_\phi)$, and we denote by q the natural map $\mathbb{P}(\mathscr{L}_\phi) \to \mathbb{P}^n$. Since $\mathscr{C}_\phi \cong i_*(\mathscr{L}_\phi)$ is a quotient of $\mathscr{B}$, we have a natural closed embedding:

$$p : \mathbb{P}(\mathscr{L}_\phi) \hookrightarrow \mathbb{P}(\mathscr{B}) = \mathcal{P}.$$

Denote by P_{Y_ϕ} the restriction to Y_ϕ of the divisor P, so that $\mathscr{O}_{Y_\phi}(P_{Y_\phi}) \cong p^*(\mathscr{O}_{\mathcal{P}}(P))$.

Lemma 2.2. *Choose any $\phi \in \mathbf{W}$ with $X_\phi \neq \emptyset$. Then the scheme $\mathbb{P}(\mathscr{L}_\phi)$ is identified with the subscheme Y_ϕ of $\mathcal{P}$, and we have:*

$$\mathscr{L}_\phi \cong q_*(\mathscr{O}_{Y_\phi}(P_{Y_\phi})), \qquad q(Y_\phi) = X_\phi. \tag{2.6}$$

Finally, assume that $\mathrm{D}_{b-2}(\phi) = \emptyset$. Then $q : \mathbb{P}(\mathscr{L}_\phi) \to \mathbb{P}^n$ is an isomorphism onto X_ϕ, so Y_ϕ is identified with X_ϕ.

Proof. The scheme $\mathbb{P}(\mathscr{L}_\phi)$ is given by pairs $([\xi], [\gamma])$ where $\xi : V \to \mathbf{k}$ represents a 1-dimensional quotient of V and the proportionality class of γ lies in $\mathbb{P}(\mathscr{L}_{\phi,\xi})$. Since $i_*(\mathscr{L}_\phi)$ is defined as $\mathrm{coker}(\phi)$, we have that γ is a quotient of $\mathscr{L}_{\phi,\xi}$ fitting into:

$$\mathscr{A}_\xi \xrightarrow{\phi_\xi} \mathscr{B}_\xi \to \mathscr{L}_{\phi,\xi}.$$

Lifting γ to a map $\mathscr{B}_\xi \to \mathbf{k}$ (still denoted by γ), we get that γ is defined on $\mathscr{L}_{\phi,\xi}$ if and only if $\gamma \circ \phi_\xi = 0$. Clearly, we have:

$$\gamma \circ \phi_\xi = 0 \Leftrightarrow \gamma(\phi_\xi(e)) = 0, \forall e \in \mathscr{A}_\xi.$$

Summing up we have:

$$\mathbb{P}(\mathscr{L}_\phi) = \{([\xi],[\gamma]) \,|\, \gamma(\phi_\xi(e)) = 0, \forall e \in \mathscr{A}_\xi\}. \tag{2.7}$$

On the other hand, Y_ϕ consists of pairs $([\xi],[\gamma])$ such that γ is a quotient of $\mathscr{B}_\xi$ and s_ϕ vanishes at $([\xi],[\gamma])$. By definition of s_ϕ, its evaluation $s_{\phi,([\xi],[\gamma])}$ at a pair $([\xi,\gamma])$ is given as the composition:

$$\mathscr{A}_\xi \xrightarrow{\phi_\xi} \mathscr{B}_\xi \xrightarrow{\gamma} \mathscr{O}_\xi(P) \cong \mathbf{k}.$$

Therefore, we have:

$$Y_\phi = \{([\xi],[\gamma]) \,|\, s_{\phi,([\xi],[\gamma])}(e) = \gamma(\phi_\xi(e)) = 0, \forall e \in \mathscr{A}_\xi\}.$$

This agrees with (2.7), so our first statement is proved.

To check (2.6), we note that, since p is induced by the projection $\mathscr{B} \to i_*(\mathscr{L}_\phi)$, there is an isomorphisms:

$$p^*(\mathscr{O}_{\mathcal{P}}(P)) \cong \mathscr{O}_{\mathscr{L}_\phi}(1).$$

Clearly we have:

$$q_*(\mathscr{O}_{\mathscr{L}_\phi}(1)) \cong \mathscr{L}_\phi.$$

This proves (2.6).

To check the last statement, first note that since $\mathscr{L}_\phi$ is supported on X_ϕ the map q takes value in X_ϕ. When $\mathrm{D}_{b-2}(\phi) = \emptyset$, the sheaf $\mathscr{L}_\phi$ is locally free of rank one on X_ϕ by [Pra88]. So $\mathbb{P}(\mathscr{L}_\phi) \cong X_\phi$. □

Note that q is the restriction to Y_ϕ of π and recall the exact sequence:

$$0 \to \mathscr{T}_\phi \to q^*(\mathscr{L}_\phi) \to \mathscr{O}_{Y_\phi}(P_{Y_\phi}) \to 0. \tag{2.8}$$

Here $\mathscr{T}_\phi$ is defined as the kernel of the canonical surjection above. It can be though of as the relative cotangent sheaf of q. This sheaf is supported on the locus in Y_ϕ which is blown down by q.

2.4. **Canonical class of a determinantal subvariety.** We define the divisor class $H_{\mathcal{P}} = \pi^*(H_{\mathbb{P}^n})$ and we consider its restriction H_{Y_ϕ} to Y_ϕ. On the variety X_ϕ, we have the hyperplane section class H_{X_ϕ}. We have defined also the class $P_{X_\phi} = q_*(P_{Y_\phi}) = c_1(\mathscr{L}_\phi)$, as en element of $\mathrm{Cl}(X)$ (see Section 2.2).

Lemma 2.3. *Let $\phi \in \mathbf{W}$ be such that $\dim(X_\phi) = n + a - b - 1$, and set:*

$$\ell = \sum_{j=1,\dots,a} \alpha_j - \sum_{i=1,\dots,b} \beta_i.$$

Then we have:

$$K_{Y_\phi} = (\ell - n - 1)H_{Y_\phi} + (a-b)P_{Y_\phi}, \tag{2.9}$$

$$c_1(\omega_{X_\phi}) = (\ell - n - 1)H_{X_\phi} + (a-b)P_{X_\phi}. \tag{2.10}$$

Proof. We first calculate the canonical class of $\mathcal{P} = \mathbb{P}(\mathscr{B})$. Since $\mathscr{B}$ is a vector bundle of rank b, this is given as:

$$\omega_{\mathcal{P}} \cong \pi^*(\omega_{\mathbb{P}^n}(c_1(\mathscr{B}))) \otimes \mathscr{O}_{\mathcal{P}}(-bP),$$

which yields:

$$K_{\mathcal{P}} = (-n-1-\sum_{i=1,\ldots,b} \beta_i)H_{\mathcal{P}} - bP.$$

Now note that Y_ϕ is a complete intersection in $\mathcal{P}$, so adjunction formula gives:

$$\begin{aligned} K_{Y_\phi} &= (K_{\mathcal{P}} + c_1(\pi^*(\mathscr{A})^*(P)))_{|Y_\phi} = \\ &= (K_{\mathcal{P}})_{|Y_\phi} + \sum_{j=1,\ldots,a} \alpha_j H_{Y_\phi} + aP_{Y_\phi} = \\ &= (\sum_{j=1,\ldots,a} \alpha_j - \sum_{i=1,\ldots,b} \beta_i - n - 1)H_{Y_\phi} + (a-b)P_{Y_\phi}. \end{aligned}$$

This proves (2.9). By Grothendieck duality, we easily obtain the isomorphism:

$$\omega_{X_\phi} \cong q_*(\omega_{Y_\phi}),$$

and by (2.9) this sheaf is isomorphic to $q_*(\mathscr{O}_{Y_\phi}((\ell-n-1)H_{Y_\phi} + (a-b)P_{Y_\phi}))$. In turn, this gives:

$$\omega_{X_\phi} \cong \mathscr{C}_\phi^{\otimes(a-b)} \otimes \mathscr{O}_{X_\phi}((\ell-n-1)H_{X_\phi}).$$

Taking the first Chern class gives (2.10). □

3. Normal sheaf of a determinantal subvariety

We consider again a morphism $\phi : \mathscr{A} \to \mathscr{B}$ and the degeneracy locus $X_\phi = \mathrm{D}_{b-1}(\phi)$. We will assume that (1.2) holds, namely:

$$\alpha_i \geq \beta_{i+1}, \quad \forall i = 1, \ldots, b-1, \quad \text{and} \quad \alpha_i > \beta_i, \quad \text{for some } i = 1, \ldots, b.$$

Proposition 3.1. *Let $X = X_\phi$ and let $\mathscr{N} = \mathscr{N}_{X,\mathbb{P}^n}$ be the normal sheaf of X in $\mathbb{P}^n$. Assume the above conditions on α_j's and β_i, $\dim(X) \geq 2$ and $c = \mathrm{codim}(X, \mathbb{P}^n) \geq 2$. Then we have:*

$$\mathrm{h}^0(X, \mathscr{N}) \leq \lambda_c + K_3 + \cdots + K_c.$$

It will turn out that the above inequality is an equality, see the proof of the main result at the end of Section 4. The proof of the above proposition will follow from the next lemmas.

Lemma 3.2. *In the numerical range (1.2), for ϕ general in* $\mathbf{W}$*, we have:* $\mathrm{codim}(\mathrm{Sing}(X_\phi), X_\phi) \geq 3$.

Proof. This follows from the result of [Cha89]. Indeed, we construct Chang's filtration of the bundles $\mathscr{B} \supset \mathscr{B}_1 \supset \cdots \supset \mathscr{B}_l$ and $\mathscr{A} \supset \mathscr{A}_1 \supset \cdots \supset \mathscr{A}_l$, as follows. We choose $\mathscr{A}_1 = \oplus_{j=1}^{r_1} \mathscr{O}_{\mathbb{P}^n}(-\alpha_j)$ with $r_1 = \max\{j | \alpha_{a+1-j} \geq \beta_b\}$ and $\mathscr{B}_1 = \oplus_{i=1}^{s_1} \mathscr{O}_{\mathbb{P}^n}(-\beta_i)$ with $s_1 = \max\{i | \beta_{b-i+1} > \alpha_{a-r_1}\}$. Iterating this procedure we get the desired filtration, and, since we are assuming $a - b + 1 \geq 2$, we obtain by the main theorem of [Cha89] that $\mathrm{codim}(X_\phi) = a - b + 1$ and $\mathrm{codim}(\mathrm{Sing}(X_\phi), X_\phi) \geq 3$. □

Lemma 3.3. *Set $\mathscr{C} = \mathscr{C}_\phi$ and $X = X_\phi$. Then, assuming* (1.1)*, we have:*

$$\mathrm{h}^0(\mathbb{P}^n, \mathscr{E}xt^1_{\mathbb{P}^n}(\mathscr{C},\mathscr{C})) \le \lambda_c + K_3 + \cdots + K_c.$$

Proof. Keep in mind that X is an integral ACM subvariety of $\mathbb{P}^n$ (call i the embedding). Since $\dim(X) \ge 2$, this implies $\mathrm{H}^1(X, \mathscr{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$. We also have $\mathrm{H}^0(X, \mathscr{O}_X) \cong \mathbf{k}$.

Further, recall that $\mathscr{C}$ is $i_*(\mathscr{L})$, where $\mathscr{L} = \mathscr{L}_\phi$ is an ACM rank-1 sheaf on X. Hence $\mathrm{H}^1(X, \mathscr{L}(t)) = 0$ for all $t \in \mathbb{Z}$. Recall the exact sequence (2.3), and rewrite it as:

$$0 \to \mathscr{O}_X \to \mathscr{B}^* \otimes \mathscr{L} \xrightarrow{\psi} \mathscr{A}^* \otimes \mathscr{L} \to \mathscr{F} \to 0, \qquad \mathscr{E}xt^1_{\mathbb{P}^n}(\mathscr{C},\mathscr{C}) \subset \mathscr{F}. \tag{3.1}$$

Then we have:

$$\mathrm{h}^0(\mathbb{P}^n, \mathscr{E}xt^1_{\mathbb{P}^n}(\mathscr{C},\mathscr{C})) \le \mathrm{h}^0(\mathscr{F}),$$

and we want to show:

$$\mathrm{h}^0(\mathscr{F}) = \lambda_c + K_3 + \cdots + K_c.$$

Let us assume for the moment that the following claims holds:

Claim 3.4. *Whenever $\dim(X) \ge 2$, we have $\mathrm{H}^1(X, \mathrm{Im}(\psi)) = 0$.*

Set $f(t) = \mathrm{h}^0(X, \mathscr{L}(t)) = \mathrm{h}^0(\mathbb{P}^n, \mathscr{C}(t))$. Once the above claim is proved, we can write:

$$\mathrm{h}^0(\mathscr{F}) = \sum_{j=1,\dots,a} f(\alpha_j) - \sum_{i=1,\dots,b} f(\beta_i) + 1. \tag{3.2}$$

In order to compute $f(t)$ we write the Buchsbaum-Rim resolution of $\mathscr{C}$. Setting $\beta = \sum_{i=1,\dots,b} \beta_i$, this reads:

$$\begin{aligned} &0 \to \wedge^a \mathscr{A} \otimes S^{c-2}\mathscr{B}^*(\beta) \to \wedge^{a-1}\mathscr{A} \otimes S^{c-3}\mathscr{B}^*(\beta) \to \cdots \\ &\cdots \to \wedge^{b+1}\mathscr{A} \otimes S^0\mathscr{B}^*(\beta) \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0. \end{aligned} \tag{3.3}$$

We set $\alpha = \sum_{j=1,\dots,a} \alpha_j$, and $\ell = \alpha - \beta$ (recall the definition of β above). Looking at the s-th term of the complex (3.3), we consider $\mathscr{E}_s = \wedge^{b+s+1}\mathscr{A} \otimes S^s(\mathscr{B}^*)(\beta)$. This gives the following expression for $f(t)$:

$$\sum_{i=1,\dots,b} \binom{n-\beta_i+t}{n} - \sum_{r=1,\dots,a} \binom{n-\alpha_r+t}{n} + \sum_{s=0}^{c-2} (-1)^s \mathrm{h}^0(\mathscr{E}_s(t)), \tag{3.4}$$

Further, it is easy to compute $\mathrm{h}^0(\mathscr{E}_s(t))$ as:

$$\sum_{\substack{1 \le i_1 < \ldots < i_{a-b-s-1} \le a \\ 1 \le j_1 \le \ldots \le j_s \le b}} \binom{n - \ell + \alpha_{i_1} + \ldots + \alpha_{i_{a-b-s-1}} + \beta_{j_1} + \cdots + \beta_{j_s} + t}{n}. \tag{3.5}$$

Note that, in view of (1.1), the term in the above binomial is strictly bounded above by $-\alpha_{b+1} + t + n$, hence the binomial vanishes for $t = \beta_i$, and $t = \alpha_{i-1}$ with $i \le b+1$. Then, combining (3.5) and (3.4), we get an expression for $f(t)$, hence for $\mathrm{h}^0(\mathscr{F})$ in view of (3.2). Recalling the definition of λ_c from (1.3) and of the K_i's from (1.5), one now easily gets the desired expression for $\mathrm{h}^0(\mathscr{F})$. □

Proof of Claim 3.4. If $\dim(X) \geq 3$ the vanishing is clear since $\mathrm{H}^1(X, \mathscr{L}(t)) = 0$ for all $t \in \mathbb{Z}$ and $\mathrm{H}^2(X, \mathscr{O}_X) = 0$. So we only have to prove the vanishing for $\dim(X) = n + b - a - 1 = 2$. Set $Y = Y_\phi$ and recall that $X = q(Y)$. The exact sequence defining $\mathrm{Im}(\psi)$ is the image via q_* of:

$$0 \to \mathscr{O}_Y \to \pi^* \mathscr{B}^* \otimes \mathscr{O}_Y(P_Y) \to (\mathcal{T}_{\mathscr{B}})_{|Y} \to 0,$$

where $\mathcal{T}_{\mathscr{B}}$ is the relative tangent bundle of π (compare with (2.5)). Then we have to prove:

$$\mathrm{H}^1(Y, (\mathcal{T}_{\mathscr{B}})|Y) = 0. \tag{3.6}$$

To obtain this vanishing, we look at the Koszul complex of s_ϕ and we tensor it by $\mathcal{T}_{\mathscr{B}}(P)$. This gives an acyclic complex whose k-th term is $\pi^*(\wedge^k \mathscr{A})(-kP) \otimes \mathcal{T}_{\mathscr{B}}(P)$. Then it suffices to show:

$$\mathrm{H}^k(\mathcal{P}, \mathcal{T}_{\mathscr{B}}((1-k)P + tH_{\mathcal{P}})) = 0, \qquad \text{for } k = 1, \ldots, a+1 \text{ and } \forall t \in \mathbb{Z}. \tag{3.7}$$

We have $a = n + b - 3$. Then, we use (2.5), after recalling that $\mathbf{R}^j \pi_*(\mathscr{O}_{\mathcal{P}}(hP))$ vanishes for $j \neq 0, b-1$, and is a wedge power of $\mathscr{B}^*$ (twisted by a line bundle) whenever it does not vanish, hence it splits as a direct sum of line bundles on $\mathbb{P}^n$. We get (3.7) for $k = 1, \ldots, a$. For $k = a+1$, we use that $\mathbf{R}^j \pi_*(\mathcal{T}_{\mathscr{B}}((-a)P))$ vanishes for $j \neq b-1$, and is a Schur functor of $\mathscr{B}^*$ for $j = b-1$. So, using $a = n+b-3$, we see that the only term contributing to $\mathrm{H}^{a+1}(\mathcal{P}, \mathcal{T}_{\mathscr{B}}(-aP + tH_{\mathcal{P}}))$ is $\mathrm{H}^{n-1}(\mathbb{P}^n, \mathbf{R}^{b-1}\pi_*(\mathcal{T}_{\mathscr{B}}(-aP))(tH)) = 0$. This concludes the proof. □

Lemma 3.5. *Assuming* (1.2)*, we have:*

$$i_*(\mathscr{N}) \cong \mathscr{E}xt^1_{\mathbb{P}^n}(\mathscr{C}, \mathscr{C}).$$

Proof. We have a natural isomorphism:

$$i_*(\mathscr{N}) \cong \mathscr{E}xt^1_{\mathbb{P}^n}(i_*(\mathscr{O}_X), i_*(\mathscr{O}_X)),$$

so we have to provide an isomorphism of the right-hand-side with $\mathscr{E}xt^1_{\mathbb{P}^n}(\mathscr{C}, \mathscr{C})$. We recall that $\mathscr{C} \cong i_*(\mathscr{L})$. In order to obtain it, we consider the cohomological spectral sequence:

$$E_2^{p,q} = \mathscr{E}xt^p_{\mathbb{P}^n}(i_*(\mathscr{O}_X), i_*(\mathscr{E}xt^q_X(\mathscr{L}, \mathscr{L}))) \Rightarrow \mathscr{E}xt^{p+q}_{\mathbb{P}^n}(\mathscr{C}, \mathscr{C}).$$

Since we have seen that $\mathscr{H}om_X(\mathscr{L}, \mathscr{L}) \cong \mathscr{O}_X$, this gives the required isomorphism once we prove $\mathscr{E}xt^1_X(\mathscr{L}, \mathscr{L}) = 0$.

In order to show this vanishing, we recall that $\mathscr{L} \cong q_*(\mathscr{O}_Y(P_Y))$, where we have set $Y = Y_\phi$. Projection formula provides a natural isomorphism:

$$\mathscr{E}xt^1_X(\mathscr{L}, \mathscr{L}) \cong \mathscr{E}xt^1_Y(q^*(\mathscr{L}), \mathscr{O}_Y(P_Y)).$$

To show that the right-hand-side vanishes, we apply $\mathscr{H}om_Y(-, \mathscr{O}_Y(P_Y))$ to the exact sequence (2.8). Clearly $\mathscr{E}xt^1_Y(\mathscr{O}_Y(P_Y), \mathscr{O}_Y(P_Y)) = 0$ for $\mathscr{O}_Y(P_Y)$ is locally free. Further, q is a birational surjective map, and the support Z_ϕ of $\mathscr{T}_\phi$ lies over $\mathrm{D}_{b-2}(\phi) \subset \mathrm{Sing}(X_\phi)$, where q is generically a $\mathbb{P}^1$-bundle. Then we have $\mathrm{codim}(Z_\phi, Y_\phi) \geq \mathrm{codim}(\mathrm{Sing}(X_\phi), X_\phi) - 1 \geq 2$ by Lemma 3.2. Then we get $\mathscr{E}xt^1_Y(\mathscr{T}_\phi, \mathscr{O}_Y(P_Y)) = 0$, so we get $\mathscr{E}xt^1_Y(q^*(\mathscr{L}), \mathscr{O}_Y(P_Y)) = 0$ and we are done. □

The previous lemmas suffice to prove Proposition (3.1).

4. Fibers of the map F

The aim of this section is to provide a proof for the main results stated in the introduction. We will let again ϕ be a morphism $\phi : \mathscr{A} \to \mathscr{B}$ such that X_ϕ has codimension $c = a - b + 1$. We will assume that (1.2) holds, namely:

$$\alpha_i \geq \beta_{i+1}, \quad \forall i = 1, \dots, b-1, \quad \text{and} \quad \alpha_i > \beta_i, \quad \text{for some } i = 1, \dots, b.$$

4.1. **Transitiveness on the fibers for fixed cokernel sheaves.** The following lemma shows that, once we fix the isomorphism class of the cokernel sheaf $\mathscr{C}_\phi$, the group $\mathbf{G}$ operates transitively on the fibers of F.

Lemma 4.1. *Assume* $\dim(X_\phi) \geq 1$ *and let* ϕ' *be a morphism* $\mathscr{A} \to \mathscr{B}$ *such that the sheaves* $\mathscr{C}_\phi$ *and* $\mathscr{C}_{\phi'}$ *are isomorphic. Then there are:*

$$g \in \mathrm{Aut}_X(\mathscr{A}), \qquad h \in \mathrm{Aut}_X(\mathscr{B}),$$

such that:

$$h \circ \phi = \phi' \circ g.$$

Proof. Clearly we have $X_{\phi'} = X_\phi$, for both schemes are the support of $\mathscr{C}_\phi$. We set $X = X_\phi$. Then, in view of Lemma 2.2, we can identify X with the image of the complete intersection subvariety $Y = Y_\phi$ of $\mathcal{P} = \mathbb{P}(\mathscr{B})$ under the map q. Further, $\mathscr{C}_\phi$ is $i_*(\mathscr{L}_\phi)$, where $\mathscr{L}_\phi$ is an ACM sheaf of rank 1 on X_ϕ, and by (2.6) we can identify $\mathscr{L}_\phi$ with $q_*(\mathscr{O}_{Y_\phi}(P_{Y_\phi}))$. Now we have:

Claim 4.2. *In the hypothesis of the lemma, we have:*

$$\mathrm{H}^1(\mathbb{P}^n, \mathrm{Im}(\phi')(t)) = 0, \qquad \textit{for all } t \in \mathbb{Z}. \tag{4.1}$$

We postpone the proof of this claim, and we assume it for the moment. Let us note that the claim amounts to $\mathrm{Ext}^1_{\mathbb{P}^n}(\mathscr{O}_{\mathbb{P}^n}(t), \mathrm{Im}(\phi)) = 0$ for all t, so that the isomorphism $\kappa : \mathscr{C}_\phi \to \mathscr{C}_{\phi'}$ lifts to a commutative diagram:

$$\begin{tikzcd} \mathscr{B} \arrow[r] \arrow[d, "\tilde{\kappa}"'] & \mathscr{C}_\phi \arrow[d, "\kappa"] \\ \mathscr{B} \arrow[r] & \mathscr{C}_{\phi'} \end{tikzcd} \tag{4.2}$$

This means that the two subschemes $Y_\phi = \mathbb{P}(\mathscr{L}_\phi)$ and $\mathbb{P}(\mathscr{L}_{\phi'})$ are identified, let us write $Y = Y_\phi$. Under this identification, the isomorphism κ is the image via q_* of an automorphism of $\mathscr{O}_Y(P_Y)$ which is thus given by a (non-zero) scalar, say λ. This lifts to $\lambda \mathrm{id}_{\mathscr{O}_{\mathcal{P}}(P)}$, and taking π_* gives back $\tilde{\kappa}$, so we can take $g = \lambda \mathrm{id}_{\mathscr{B}}$. We have thus a diagram:

$$\begin{tikzcd} \pi^*(\mathscr{A}) \arrow[r, "s_\phi"] \arrow[d, dotted, "\mu"'] & \mathscr{O}_{\mathcal{P}}(P) \arrow[d, "\lambda"] \\ \pi^*(\mathscr{A}) \arrow[r, "s_{\phi'}"] & \mathscr{O}_{\mathcal{P}}(P) \end{tikzcd} \tag{4.3}$$

and we would like to prove the existence of a lifting μ. This is guaranteed if we show:

$$\mathrm{Ext}^1_{\mathcal{P}}(\pi^*(\mathscr{A}), \mathscr{S}_\phi) \cong \mathrm{H}^1(\mathcal{P}, \mathscr{S}_\phi \otimes \pi^*(\mathscr{A})^*) = 0,$$

where $\mathscr{S}_\phi$ is the kernel of $s_\phi : \pi^*(\mathscr{A}) \to \mathscr{O}_{\mathcal{P}}(P)$, i.e. the first syzygy arising in the Koszul complex associated to s_ϕ. In fact, we will prove the following stronger:

Claim 4.3. *In the hypothesis of the lemma, we have, for all $t \in \mathbb{Z}$:*

$$\mathrm{H}^1(\mathcal{P}, \mathscr{S}_\phi(tH_{\mathcal{P}})) = 0. \tag{4.4}$$

Let us assume the claim and show that it allows to finish the proof. We first have to prove that μ is an automorphism of $\pi^*(\mathscr{A})$. To do so, we choose a basis $e_1, \ldots, e_a$ of the decomposable bundle $\mathscr{A}$, in such a way that the morphism μ is represented in this basis by an upper triangular matrix:

$$N_\mu = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \cdots \\ 0 & \lambda_{2,2} & \lambda_{2,3} & \cdots \\ \vdots & 0 & \ddots & \end{pmatrix},$$

where for all $j = 1, \ldots, a$ the $\lambda_{j,j}$'s are scalars. In this basis, the sections s_ϕ and $s_{\phi'}$ are given by two a-tuples $(f_1, \ldots, f_a)$ and $(f'_1, \ldots, f'_a)$ which form regular sequences defining Y_ϕ. Clearly μ is an isomorphism if and only if all the $\lambda_{j,j}$'s are non-zero. But, if μ is not an isomorphism, we let j_0 be the first index j such that $\lambda_{j,j} = 0$. Then, Diagram (4.3) says that the equation f_{j_0} of Y_ϕ lies in the ideal generated by $(f'_1, \ldots, f'_{j_0-1})$, which is equal to the ideal generated by $(f_1, \ldots, f_{j_0-1})$ (since $\lambda_{j,j} \neq 0$ for $j < j_0$). This contradicts that $(f_1, \ldots, f_a)$ is a regular sequence.

We have thus proved the existence of an isomorphisms μ fitting into (4.3). Pushing this diagram to $\mathbb{P}^n$ by π gives a commutative diagram:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\phi"] \arrow[d, "\pi_*(\mu)"'] & \mathscr{B} \arrow[d, "\lambda \mathrm{id}_{\mathscr{B}}"] \\ \mathscr{A} \arrow[r, "\phi'"] & \mathscr{B} \end{tikzcd}$$

Thus we can take $h = \pi_*(\mu)$ and the lemma is proved. $\square$

Proof of Claim 4.2 and 4.3. We prove both claims by descending induction on the codimension $c = a - b + 1$ of X_ϕ. In order to do so, we add a new term $\mathscr{O}_{\mathbb{P}^n}(-\beta_{b+1})$ to $\mathscr{B}$ to obtain a new bundle $\mathscr{B}_{b+1} = \mathscr{B} \oplus \mathscr{O}_{\mathbb{P}^n}(-\beta_{b+1})$. For consistency of notation we write $\mathscr{B} = \mathscr{B}_b$, $X_\phi = X_b$, $\mathcal{P}_b = \mathbb{P}(\mathscr{B}_b)$, $P_b = P_{\mathcal{P}_b}$, etc. This process can be iterated for all $r \leq a - b$ hence constructing X_{b+r}, $\mathcal{P}_{j+r}$, Y_{b+r}, etc. At each step, we take care in adding $\mathscr{O}_{\mathbb{P}^n}(-\beta_{b+r})$ so that there exists a morphism $\phi_{b+r} : \mathscr{A} \to \mathscr{B}_{b+r}$ such that X_{b+r} is of codimension $a - b - r + 1$. That is, we add the β_{b+r} so that (1.2) is still satisfied. Set $s = b + r$, so $b \leq s \leq a$ and $\mathrm{codim}(X_s, \mathbb{P}^n) = 1$ iff $s = a$.

In order to prove Claim 4.3, we need some exact sequences of sheaves on our projective bundles $\mathcal{P}_s$. First, we have the obvious exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-\beta_s) \to \mathscr{B}_s \to \mathscr{B}_{s-1} \to 0,$$

inducing an inclusion $\mathcal{P}_{s-1} \subset \mathcal{P}_s$. We have the defining exact sequences:

$$0 \to \mathscr{O}_{\mathcal{P}_s}(-\beta_s H_s - P_s) \to \mathscr{O}_{\mathcal{P}_s} \to \mathscr{O}_{\mathcal{P}_{s-1}} \to 0,$$

and restricting to Y_s we get:

$$0 \to \mathscr{O}_{Y_s}(-\beta_s H_s - P_s) \to \mathscr{O}_{Y_s} \to \mathscr{O}_{Y_{s-1}} \to 0. \tag{4.5}$$

This gives the exact sequence:

$$0 \to \mathcal{I}_{Y_s,\mathcal{P}_s}(-\beta_s H_s) \to \mathcal{I}_{Y_s,\mathcal{P}_s}(P_s) \xrightarrow{\zeta} \mathcal{I}_{Y_{s-1},\mathcal{P}_{s-1}}(P_{s-1}) \to 0. \tag{4.6}$$

The section $s_{\phi_s} : \pi_s^*(\mathscr{A}) \to \mathscr{O}_{\mathcal{P}_s}(P_s)$ restricts to $s_{\phi_{s-1}}$ on $\mathcal{P}_{s-1}$, so we get an exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{S}_s \arrow[r] \arrow[d, dotted, "\varsigma"'] & \pi_s^*(\mathscr{A}) \arrow[r, "s_\phi"] \arrow[d, equal] & \mathcal{I}_{Y_s,\mathcal{P}_s}(P_s) \arrow[r] \arrow[d, "\zeta"] & 0 \\
0 \arrow[r] & \mathscr{S}'_{s-1} \arrow[r] & \pi_s^*(\mathscr{A}) \arrow[r, "s_{\phi'}"] & \mathcal{I}_{Y_{s-1},\mathcal{P}_{s-1}}(P_{s-1}) \arrow[r] & 0,
\end{tikzcd}$$

where the sheaf $\mathscr{S}'_{s-1}$ defined by the above diagram fits into the exact sequence:

$$0 \to \pi_s^*\mathscr{A}(-\beta_s H_s - P_s) \to \mathscr{S}'_{s-1} \to \mathscr{S}_{s-1} \to 0. \tag{4.7}$$

Finally note that, applying snake lemma to the above diagram and using (4.6) we get:

$$0 \to \mathscr{S}_s \xrightarrow{\varsigma} \mathscr{S}'_{s-1} \to \mathcal{I}_{Y_s,\mathcal{P}_s}(-\beta_s H_s) \to 0. \tag{4.8}$$

Let us now prove the desired vanishing (4.4). Let us first look at the case $s = a$, i.e. $\operatorname{codim}(X_s, \mathbb{P}^n) = 1$. We have a spectral sequence:

$$E_2^{i,j} = \mathrm{H}^i(\mathbb{P}^n, \mathbf{R}^j\pi_*(\mathscr{S}_s)(t)) \Rightarrow \mathrm{H}^{i+j}(\mathcal{P}, \mathscr{S}_s(tH_s)).$$

Applying $\mathbf{R}\pi_*$ to the exact sequence:

$$0 \to \mathscr{S}_s \to \pi^*\mathscr{A} \to \mathscr{O}_{\mathcal{P}}(P_s) \to \mathscr{O}_{Y_s}(P_s) \to 0,$$

we obtain, $\pi_*(\mathscr{O}_{Y_s}(P_s)) \cong \mathscr{C}_s$ and, for $s = a$, an exact sequence:

$$0 \to \mathscr{A} \to \mathscr{B}_s \to \mathscr{C}_s \to 0.$$

So $\mathbf{R}^j\pi_*(\mathscr{S}_s) = 0$ for all j and the vanishing is proved.

Let us now look at the induction step. In view of (4.7) and (4.8) the vanishing (4.4) will be ensured for $s = a-1, \ldots, b$ if we prove, for all $t \in \mathbb{Z}$:

$$\mathrm{H}^2(\mathcal{P}_s, \pi_s^*\mathscr{A}(tH_s - P_s)) = 0, \tag{4.9}$$

$$\mathrm{H}^1(\mathcal{P}_s, \mathcal{I}_{Y_s,\mathcal{P}_s}(tH_s)) = 0, \tag{4.10}$$

$$\mathrm{H}^1(\mathcal{P}_s, \mathscr{S}_s(tH_s)) = 0,$$

where the last vanishing holds by induction hypothesis.

The vanishing (4.9) is immediate in view of projection formula, since $\mathbf{R}^j\pi_*(\mathscr{O}_{\mathcal{P}_s}(-P_s)) = 0$ for all j. It remains only to check (4.10). This will be proved once we show that the map:

$$\mathrm{H}^0(\mathcal{P}_s, \mathscr{O}_{\mathcal{P}_s}(tH_s)) \to \mathrm{H}^0(Y_s, \mathscr{O}_{Y_s}(tH_s))$$

is surjective for all $t \geq 0$. By taking π_*, this amounts to ask that the map:

$$\mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(t)) \to \mathrm{H}^0(X_s, \mathscr{O}_{X_s}(t))$$

is surjective for all $t \geq 0$. But this fact holds since X_s is an ACM variety, see for instance [BV88], so Claim (4.3) is proved.

Let us now turn to Claim (4.2). Just to keep continue with the same notations, we prove it for ϕ rather than ϕ'. The exact sequence (4.5), twisted by P_s induces:

$$0 \to \mathscr{O}_{X_s}(-\beta_s) \to \mathscr{C}_s \to \mathscr{C}_{s-1} \to 0,$$

with obvious notation. Hence, setting $I_s = \mathrm{Im}(\phi_s)$ we have:

$$0 \to \mathcal{I}_{X_s}(-\beta_s) \to I_s \to I_{s-1} \to 0.$$

For $s = a$ the vanishing (4.1) is clear for $I_a \cong \mathscr{A}$. Moreover, recall again that X_s is ACM, so $\mathrm{H}^2(\mathbb{P}^n, \mathcal{I}_{X_s}(t)) = 0$ for all $t \in \mathbb{Z}$ whenever $\dim(X_s) \geq 2$. Taking cohomology of the exact sequence above, we see that this implies $\mathrm{H}^1(\mathbb{P}(V), I_{s-1}(t)) = 0$ for all $s \geq 1$ (so that $\dim(X_s) \geq 2$). The vanishing (4.1) is thus proved. □

4.2. **Finiteness and uniqueness of determinantal representations.** We start with two lemmas that account for the finiteness and the uniqueness of the fiber of the map F, i.e. of determinantal representations of a given subvariety $X = X_\phi$ of $\mathbb{P}^n$. This will lead to the proof of our main result.

Lemma 4.4. *Assume:*

$$c = a - b + 1 \geq 2, \qquad \dim(X_\phi) = n - c \geq 1.$$

Then, up to the action of $\mathbf{G}$*, there are finitely many elements* $\phi' \in \mathbf{W}$ *such that:*

$$\mathscr{C}_\phi \not\cong \mathscr{C}_{\phi'}, \qquad X_\phi = X_{\phi'}.$$

Proof. Let ϕ' be a morphism $\mathscr{A} \to \mathscr{B}$ such that $X_\phi = X_{\phi'}$ and set $X = X_\phi$. Then ϕ' defines a cokernel sheaf $\mathscr{C}_{\phi'} \cong i_*(\mathscr{L}_{\phi'})$. Also, X is the image via a map q' of $Y_{\phi'} = \mathbb{V}(s_{\phi'})$, according to Lemma 2.2, and we have $\mathscr{L}_{\phi'} \cong q_*(\mathscr{O}_{Y_{\phi'}}(P'_{Y_{\phi'}}))$. We set $c_1(\mathscr{L}_{\phi'}) = P'_X$, again as an element of $\mathrm{Cl}(X)$.

We can thus apply Lemma 2.3, which gives:

$$(\ell - v)H_X + (a-b)P_X = (\ell - v)H_X + (a-b)P'_X, \qquad \text{in } \mathrm{Cl}(X).$$

Pulling back to Y_ϕ we get the equality,

$$(a-b)(P_{Y_\phi} - q^*(P'_X)) = 0, \qquad \text{in } \mathrm{Cl}(Y_\phi).$$

We observe that q^* gives an isomorphism between $\mathrm{Cl}(Y_\phi)$ and $\mathrm{Cl}(X)$, for q is biregular outside a closed subset of codimension at least 2. Moreover, P_ϕ is a Cartier divisor of Y_ϕ so the above equality takes place in $\mathrm{Pic}(Y_\phi)$. Note that $a - b \neq 0$ by hypothesis, and that $\mathrm{Pic}(Y_\phi)$ has only finitely many points of order $(a-b)$. Then, recalling that $c_1(\mathscr{L}_{\phi'})$ determines $\mathscr{L}_{\phi'}$ (see Section 2.2), we get that there are only finitely many ways to choose the isomorphism class of $\mathscr{L}_{\phi'}$ in such a way that the above equation is satisfied.

In other words, there are $\phi_1, \dots, \phi_r$ such that $\mathscr{C}_{\phi_i} \not\cong \mathscr{C}_{\phi_j}$ if $i \neq j$, and such that for any other $\phi_0 \in \mathbf{W}$ we have $\mathscr{C}_{\phi_0} \cong \mathscr{C}_{\phi_i}$ for one $i = 1, \dots, r$. By Lemma (4.1), this ϕ_0 is taken to ϕ_i by the action of $\mathbf{G}$, which proves our claim. □

Lemma 4.5. *Let ϕ be a general element of* $\mathrm{Hom}(\mathscr{A},\mathscr{B})$*, and assume:*

$$\beta_b < \alpha_1, \qquad \dim(X_\phi) \geq 2, \qquad c = a - b + 1 \geq 2.$$

Then, given another morphism ϕ' with $X_\phi = X_{\phi'}$, we have $\mathscr{C}_\phi \cong \mathscr{C}_{\phi'}$.

Proof. Let $X = X_\phi$, and consider the divisor P_X associated to $\mathscr{C}_\phi$. In view of the considerations of the previous lemma, any ϕ' such that $X_{\phi'} = X$ gives a divisor class P'_X such that:

$$(a-b)(P_{Y_\phi} - q^*(P'_X)) = 0, \qquad \text{in } \mathrm{Pic}(Y_\phi).$$

Again $(a-b) \neq 0$, so that $P_{Y_\phi} - q^*(P'_X)$ is of order $(a-b)$ in $\mathrm{Pic}(Y_\phi)$. Now Y_ϕ is isomorphic to a complete intersection in $\mathcal{P}$, by a divisors of classes:

$$(\alpha_1 H_\mathcal{P} + P_\mathcal{P}, \ldots, \alpha_a H_\mathcal{P} + P_\mathcal{P}).$$

Under the hypothesis $\beta_b < \alpha_1$, all the divisors $\alpha_j H_\mathcal{P} + P_\mathcal{P}$ are very ample on $\mathcal{P}$, indeed the direct image $\pi_*(\mathscr{O}_\mathcal{P}(\alpha_j H_\mathcal{P} + P_\mathcal{P}))$ decomposes as a direct sum of positive line bundles. Therefore we can argue, by Grothendieck-Lefschetz theorem, that Y_ϕ is smooth and $\mathrm{Pic}(Y_\phi)$ is torsion-free provided that $\dim(Y_\phi) = \dim(X) \geq 2$ (see for instance [Bǎd78]). Then $q^*(P'_X) = P_{Y_\phi}$ hence $\mathscr{C}_\phi \cong \mathscr{C}_{\phi'}$ as in the previous proof. □

Proof of the main Theorem. The map F is defined on an dense open subset of $\mathcal{Y}$ as soon as there is $\phi \in \mathbf{W}$ such that X_ϕ has codimension $(a-b+1)$, and this is ensured by (1.1).

By hypothesis we have $\dim(X_\phi) = n + b - a - 1 \geq 1$, so we can apply Lemma 4.4. Applying this lemma, we get that there are only finitely many $\mathbf{G}$-orbits in the inverse image of a given point $[X_\phi] \in \mathcal{H}$, so that F is generically finite, which is (i).

Moreover, assuming $\dim(X_\phi) = n + b - a - 1 \geq 2$, by Proposition 3.1 and Lemma 2.1 we have $\dim \mathcal{T}_{[X_\phi],\mathcal{H}} \leq \dim(\mathcal{Y})$, and $\mathcal{Y}$ is irreducible and generically smooth.

Since F is generically finite, the image of F also has dimension $\dim(\mathcal{Y})$, so $\dim \mathcal{T}_{[X_\phi],\mathcal{H}} = \dim(\mathcal{H})$ (i.e. the inequality in Proposition (3.1) is an equality). Hence $\mathcal{H}$ is generically smooth and F is dominant on $\mathcal{H}$, so $\mathcal{H}$ is also irreducible. This proves (ii).

Finally, assuming $\beta_b < \alpha_1$ and $\dim(X_\phi) = n + b - a - 1 \geq 2$, $c = a - b + 1 \geq 2$, we can apply Lemma 4.5. Then, by Lemma 4.1 the action of $\mathbf{G}$ is generically transitive on the set of determinantal representations of X_ϕ, so F is generically injective. Still F is dominant by part (ii), hence F is birational. □

5. Further remarks on the geometry of determinantal subvarieties

In this section we carry out some remarks on the geometry of determinantal submvarieties, mainly focused on determinantal surfaces and hypersurfaces.

5.1. **Determinantal surfaces.**

We collect here some simple examples of determinantal surfaces accounting for different behavior of the Picard group.

Proposition 5.1. *The Picard number of a smooth determinantal surface can be arbitrarily high if $\ell < n+1$.*

On the other hand this number is 2 if $\mathbf{k} = \mathbb{C}$, $\beta_b < \alpha_1$, $a > b$ and $\ell > n+1$.

Proof. We give two examples of the first phenomenon.

(1) Let $n \geq 3$, and let ϕ be a general morphism:
$$\mathscr{O}_{\mathbb{P}^n}(-1)^n \to \mathscr{O}_{\mathbb{P}^n}^3.$$
Then, according to the construction of Section 2.3, the smooth surface X_ϕ is the blow up of $\mathbb{P}^2$ at the collection Z of $\binom{n+1}{2}$ general points, embedded by the linear system of curves of degree n passing through them. Indeed, Lemma 2.2 gives that X_ϕ is $\mathbb{P}(\mathcal{I}_Z)$, where the ideal sheaf $\mathcal{I}_Z$ fits in:
$$0 \to \mathscr{O}_{\mathbb{P}^2}^n(-1) \to \mathscr{O}_{\mathbb{P}^2}^{n+1} \to \mathcal{I}_Z(n) \to 0.$$

(2) Let $n \geq 3$, and let ϕ be a general morphism:
$$\mathscr{O}_{\mathbb{P}^n}(-1)^{n-2} \oplus \mathscr{O}_{\mathbb{P}^n}(-2) \to \mathscr{O}_{\mathbb{P}^n}^2.$$
Then we claim that X_ϕ is the blow up of $\mathbb{P}^2$ at $2n$ general points. More precisely, if we let P be pull-back to X_ϕ of the class of a line in $\mathbb{P}^2$, and by $E_1, \ldots, E_{2n}$ the exceptional divisor, then the linear system H_{X_ϕ} is given by:
$$nP - (n-2)E_1 - E_2 - \cdots - E_{2n}.$$
Indeed, one easily sees that such linear system provides the above determinantal representation. On the other hand, counting parameters of these blown-up planes, and using our main theorem we get that a general ϕ degenerates along a blown-up plane.

For the second statement, we borrow the notation from the previous section. We note that under the hypothesis $\beta_b < \alpha_1$, we have that $P_{\mathcal{P}} + \alpha_j H_{\mathcal{P}}$ is a very ample divisor on $\mathcal{P}$ for any $j = 1, \ldots, a$, and Y_ϕ is a complete intersection of these divisors. Hence $\mathrm{Pic}(Y_\phi) \cong \mathrm{Pic}(\mathcal{P}) \cong \mathbb{Z}^2$ as long as $\dim(Y_\phi) \geq 3$ and ϕ is general (for $\mathbf{k}$ algebraically closed of any characteristic). In this case, since X is ACM, we also get $\mathrm{H}^2(Y_\phi, \mathscr{O}_{Y_\phi}) = \mathrm{H}^2(X, \mathscr{O}_X) = 0$, so $\mathrm{h}^{2,0}(Y_\phi) = 0$. If moreover $\ell > n+1$ and $\dim(Y_\phi) = 2$ then $\mathrm{h}^{2,0}(Y_\phi) = \mathrm{h}^0(Y_\phi, \omega_{Y_\phi})$, and we easily see that $\mathrm{h}^0(Y_\phi, \omega_{Y_\phi}) \neq 0$. So by [Moĭ67] we get $\mathrm{Pic}(Y_\phi) \cong \mathbb{Z}^2$. $\square$

5.2. **Hypersurfaces.** A determinantal hypersurface is necessarily singular as soon as $n \geq 4$. However we have the following result.

Proposition 5.2. *Up to conjugacy, a hypersurface in $\mathbb{P}^n$ has finitely many (minimal) determinantal representations if $n \geq 4$.*

Moreover, a general hypersurface in $\mathbb{P}^n$, with $n \geq 4$, and $d \geq 4$ if $n = 3$, has a unique determinantal representations (up to conjugacy) if $\mathbf{k} = \mathbb{C}$, $\beta_b < \alpha_1$, $\ell > n+1$.

Proof. Let f be an equation of the surface X under consideration. First note that there are finitely many ways to choose $\mathscr{A}$ and $\mathscr{B}$ in such a way that a minimal morphism $\phi : \mathscr{A} \to \mathscr{B}$ (i.e. with no non-zero terms of degree zero) has $\det(\phi) = f$.

Then, for any such choice of $\mathscr{A}$ and $\mathscr{B}$, by Lemma 4.1 the $\mathbf{G}$-orbits (i.e. the conjugacy classes) are classified by the cokernel sheaves $\mathscr{C}_\phi$ up to isomorphism. Any such cokernel sheaf is the direct image on X of the line bundle $\mathscr{O}_Y(P)$ on the variety $Y = Y_\phi$ by the map π_Y (see the notation of the preceding Section). All of these line

bundles are rigid in view of $\mathrm{H}^1(Y, \mathscr{O}_Y) = \mathrm{H}^1(X, \mathscr{O}_X) = 0$. Therefore there cannot be infinitely many choices for $\mathscr{C}_\phi$, nor for the conjugacy class of ϕ.

For the second part of the proposition, we would like to show that $\mathscr{C}_\phi$ is uniquely determined by X, or equivalently to show that the line bundle $\mathscr{O}_Y(P)$ is determined by X. If $\beta_b < \alpha_1$, again $P_{\mathcal{P}} + \alpha_j H_{\mathcal{P}}$ is a very ample divisor on $\mathcal{P}$ for any $j = 1, \ldots, a$, so $\mathrm{Pic}(Y) \cong \mathbb{Z}^2$ as long as $\dim(Y) \geq 3$ (for $\mathbf{k}$ algebraically closed of any characteristic). If moreover $\ell > n + 1$ and $\dim(Y) = 2$ then $\mathrm{h}^{2,0}(X) \neq 0$, as in the proof of the previous proposition, so $\mathrm{Pic}(X) \cong \mathbb{Z}^2$, generated by the classes H_{Y_ϕ}, P_{Y_ϕ}, and these generators are clearly distinguished by their number of global sections. Thus $\mathcal{P}_{Y_\phi}$ is determined by Y_ϕ. $\square$

Remark 5.3. We note that finiteness of conjugacy classes of determinantal plane curves C of genus ≥ 1 definitely fails. Indeed, these classes are birationnally classified by the Jacobian of C see [Bea00, Cat81], a variety of positive dimension.

Remark 5.4. Note that uniqueness of determinantal representations does take place for surfaces in $\mathbb{P}^3$ of degree ≤ 3. For instance, according to [Fae08], the possible choices of the α_j's and β_i's for a cubic surface are:

(1) $\alpha_1 = 1, \alpha_2 = 2$, $\beta_1 = \beta_2 = 0$, this gives 27 non-conjugate ϕ's;
(2) $\alpha_1 = \alpha_2 = 2$, $\beta_1 = 1, \beta_2 = 0$, this gives 27 more non-conjugate ϕ's;
(3) $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = \beta_3 = 0$, giving 72 non-conjugate ϕ's.

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